PERGAMON

International Journal of Solids and Structures 38 (2001) 2921–2931

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A thermodynamically consistent approach to microplane theory. Part I. Free energy and consistent microplane stresses

Ignacio Carol [a,*], Milan Jirásek [b], Zdeněk Bažant [c]

[a] *School of Civil Engineering (ETSECCPB), Technical University of Catalonia (UPC), E–08034 Barcelona, Spain*
[b] *Laboratory of Structural and Continuum Mechanics, Swiss Federal Institute of Technology at Lausanne (EPFL), CH-1015 Lausanne, Switzerland*
[c] *Department of Civil Engineering, Northwestern University, Evanston, IL 60208, USA*

Received 12 June 1999; in revised form 16 April 2000

Abstract

Microplane models are based on the assumption that the constitutive laws of the material may be established between normal and shear components of stress and strain on planes of generic orientation (so-called microplanes), rather than between tensor components or their invariants. In the kinematically constrained version of the model, it is assumed that the microplane strains are projections of the strain tensor, and the stress tensor is obtained by integrating stresses on microplanes of all orientations at a point. Traditionally, microplane variables were defined intuitively, and the integral relation for stresses was derived by application of the principle of virtual work. In this paper, a new thermodynamic framework is proposed. A free-energy potential is defined at the microplane level, such that its integral over all orientations gives the standard macroscopic free energy. From this simple assumption, it is possible to introduce consistent microplane stresses and their corresponding integral relation to the macroscopic stress tensor. Based on this, it is shown that, in spite of the excellent data fits achieved, many existing formulations of microplane model were not guaranteed to be fully thermodynamically compliant. A consequence is the lack of work conjugacy between some of the microplane stress and strain variables used, and the danger of spurious energy dissipation/generation under certain load cycles. The possibilities open by the new theoretical framework are developed further in Part II companion paper.

Keywords: Microplane model; Constitutive model; Thermodynamics; Anisotropy; Concrete; Damage; Plasticity

1. Introduction

Since it was first proposed by Bažant and Oh (1983), the microplane approach has become progressively more popular for the description of the constitutive behavior of a number of engineering materials such as concrete, rock, ceramics, or ice (Bažant and Gambarova, 1984; Bažant and Oh, 1985; Bažant and Prat, 1987, 1988a,b; Carol et al., 1991, 1992; Cofer, 1992; Ožbolt and Bažant, 1992; Cofer and Kohut, 1994; Bažant et al., 1996a,b, 2000a,b; Fichant, 1996; Ožbolt and Bažant, 1996; Kuhl et al., 1998).

* Corresponding author. Fax: +34-93-401-7251.
E-mail address: ignacio.carol@upc.es (I. Carol).

0020-7683/01/$ - see front matter
PII: S0020-7683(00)00212-2

The main idea behind microplane models consists in developing the constitutive laws for the two- or three-dimensional continuum starting from the behavior of a plane of generic orientation, which is then integrated over all possible directions in space. This idea is actually not new. The classical elasto-plastic failure envelopes such as Tresca and Mohr-Coulomb may be also derived from the idea of a limit σ–τ condition for a generic plane (Mohr, 1900). The slip theory of plasticity (Taylor, 1938; Batdorf and Budiansky, 1949) and the viscoplastic multilaminate model for fractured rocks and soils (Zeinkiewicz and Pande, 1977; Pande and Sharma, 1983) were also based on similar concepts. The main difference between the microplane model and the previous similar models is the kinematic constraint assumed, and the principle of virtual work (PVW) applied to obtain the corresponding integral micro–macro relation for stresses. This is well documented in the literature, (Carol and Bažant, 1997), and is only briefly summarized in Section 2.

Although successfully implemented and extensively verified with experimental results (Bažant and Prat, 1988b; Carol et al., 1992, Bažant et al., 1996b, Bažant et al., 2000a) the traditional microplane models were to some extent based on intuitive arguments, and their thermodynamic consistency could not be guaranteed in all loading situations. It turns out that, in the way they have been introduced, some of the stress variables used at the microplane level are not conjugate quantities to their strain counterparts. The lack of full thermodynamic consistency (actually common to many constitutive models used in engineering practice) seems to have had little influence on the representation of available experimental data, given the excellent fits obtained under numerous different loading conditions. But no doubt there must exist load sequences for which energy is spuriously dissipated or generated and could be large enough to distort the predicted material response. In any case, it is obvious that an approach in which conjugacy of variables and thermodynamical consistency is assured should always be preferable.

A first simple version of such a consistent approach with conjugate volumetric and deviatoric microplane stresses has recently been proposed in the context of the extension of microplane theory to finite deformations (Carol et al., 1998), and is now developed in some detail for small strain in the form of a two-part paper. In this first part, the new fundamental assumption of a free-energy potential and the most immediate consequences, such as definition of consistent microplane stresses and integral micro–macro relations, are developed. This is done in Section 3. In Section 4, the resulting integral formula is compared to the traditional expressions presented in Section 2. Differences are discussed and interpreted, highlighting under what conditions both formulations could be considered equivalent. Section 5 presents an example of a specific microplane formulation that can lead to spurious energy dissipation or generation. Finally, the first set of conclusions is given in Section 6.

2. Traditional derivation of microplane models via PVW

Microplane models construct the "macroscopic" response of the material (constitutive laws relating the tensors of stress and strain) from the cumulative effect of processes taking place on elementary planes of different orientations called microplanes. The orientation of each microplane is described by the unit normal vector, $\mathbf{n}$. The deformation and stresses on the microplane are characterized by the normal and shear strains, ε_N, $\boldsymbol{\varepsilon}_T$ (with Cartesian components ε_{T_r}), and the corresponding microplane tractions, σ_N, and $\boldsymbol{\sigma}_T$ (with Cartesian components σ_{T_r}). With the exception of the earliest formulations (Bažant and Oh, 1983; Bažant and Gambarova, 1984), which worked very well for distributed multidirectional tensile cracking but could not cope with the nonlinearity under compression and shear, most versions of the model assume the normal components to be further split into their volumetric and deviatoric parts, ε_V and ε_D (or σ_V and σ_D). The *kinematic constraint* means that the normal and shear strains on the microplane are assumed equal to the projections of the macroscopic strain tensor ε_{ij} (as opposite to a static constraint in previous models based on similar ideas):

$$\begin{aligned}&\varepsilon_{\mathrm{N}} = n_i n_j \varepsilon_{ij} \quad (\text{or, with split, } \varepsilon_{\mathrm{V}} = \frac{\delta_{ij}}{3}\varepsilon_{ij},\ \varepsilon_{\mathrm{D}} = \varepsilon_{\mathrm{N}} - \varepsilon_{\mathrm{V}}),\\ &\varepsilon_{\mathrm{T}_r} = \varepsilon_{rj}n_j - \varepsilon_{\mathrm{N}}n_r = \tfrac{1}{2}\varepsilon_{ir}n_i + \tfrac{1}{2}\varepsilon_{rj}n_j - n_i n_j \varepsilon_{ij} n_r = \tfrac{1}{2}[n_i\delta_{jr} + n_j\delta_{ir} - 2n_i n_j n_r]\varepsilon_{ij},\end{aligned} \tag{1}$$

where the lowercase subscripts in latin refer to Cartesian coordinates x_i ($i = 1,2,3$), and subscript repetition implies summation. The same relations may be expressed in compact notation as

$$\begin{aligned}&\varepsilon_{\mathrm{N}} = \boldsymbol{N} : \boldsymbol{\varepsilon} \quad (\text{or, with split, } \varepsilon_{\mathrm{V}} = \boldsymbol{V} : \boldsymbol{\varepsilon},\ \varepsilon_{\mathrm{D}} = \boldsymbol{D} : \boldsymbol{\varepsilon}),\\ &\boldsymbol{\varepsilon}_{\mathrm{T}} = \boldsymbol{T} : \boldsymbol{\varepsilon},\end{aligned} \tag{2}$$

where the projection tensors $\boldsymbol{N}$, $\boldsymbol{V}$, $\boldsymbol{D}$ (of second order) and $\boldsymbol{T}$ (of third order) have the Cartesian components,

$$\begin{aligned}&N_{ij} = n_i n_j, \qquad V_{ij} = \frac{\delta_{ij}}{3},\\ &D_{ij} = n_i n_j - \frac{\delta_{ij}}{3}, \qquad T_{rij} = \frac{1}{2}[n_i\delta_{jr} + n_j\delta_{ir} - 2n_i n_j n_r].\end{aligned} \tag{3}$$

Material laws are constructed at the microplane level in the form of functions,

$$\begin{aligned}&\sigma_{\mathrm{N}} = \mathscr{F}_{\mathrm{N}}(\varepsilon_{\mathrm{N}}) \quad (\text{or, with split, } \sigma_{\mathrm{V}} = \mathscr{F}_{\mathrm{V}}(\varepsilon_{\mathrm{V}}),\ \sigma_{\mathrm{D}} = \mathscr{F}_{\mathrm{D}}(\varepsilon_{\mathrm{D}})),\\ &\boldsymbol{\sigma}_{\mathrm{T}} = \boldsymbol{F}_{\mathrm{T}}(\boldsymbol{\varepsilon}_{\mathrm{T}}, \varepsilon_{\mathrm{V}}).\end{aligned} \tag{4}$$

With the kinematic constraint and *general* microplane material laws, equilibrium between the macro and microstresses is not possible in a ‘strong’ sense (i.e., the static constraint dual to Eq. (1) is *not* satisfied). The *weak* form of micro–macro equilibrium equations can be constructed using the principle of virtual work,

$$\frac{4\pi}{3}\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} = 2\int_{\Omega}[\sigma_{\mathrm{N}}\,\delta\varepsilon_{\mathrm{N}} + \boldsymbol{\sigma}_{\mathrm{T}}\cdot\delta\boldsymbol{\varepsilon}_{\mathrm{T}}]\,\mathrm{d}\Omega, \tag{5}$$

where Ω is the surface of a unit hemisphere (representing the set of all possible microplane orientations). Substituting $\delta\varepsilon_{\mathrm{N}} = \boldsymbol{N} : \delta\boldsymbol{\varepsilon}$ and $\delta\boldsymbol{\varepsilon}_{\mathrm{T}} = \boldsymbol{T} : \delta\boldsymbol{\varepsilon}$ and taking account of the independence of individual components of the (symmetric) virtual strain tensor, we get the integral micro–macro equilibrium condition,

$$\boldsymbol{\sigma} = \frac{3}{2\pi}\int_{\Omega}\sigma_{\mathrm{N}}\boldsymbol{N}\,\mathrm{d}\Omega + \frac{3}{2\pi}\int_{\Omega}\boldsymbol{\sigma}_{\mathrm{T}}\cdot\boldsymbol{T}\,\mathrm{d}\Omega \tag{6}$$

or in index notation

$$\sigma_{ij} = \frac{3}{2\pi}\int_{\Omega}\sigma_{\mathrm{N}}n_i n_j\,\mathrm{d}\Omega + \frac{3}{2\pi}\int_{\Omega}\frac{\sigma_{\mathrm{T}_r}}{2}\left[n_i\delta_{rj} + n_j\delta_{ri}\right]\mathrm{d}\Omega \tag{7}$$

(when rewriting the second integral, the last term in the definition of $\boldsymbol{T}$ (Eq. (3), fourth term) has been omitted because $\boldsymbol{\sigma}_{\mathrm{T}}$ is a vector contained in the microplane and therefore $\sigma_{\mathrm{T}_r}n_r = 0$).

In previous models with volumetric-deviatoric split similar to (Bažant and Prat, 1988a), σ_{N} in the first term of the previous equation was directly replaced by $\sigma_{\mathrm{V}} + \sigma_{\mathrm{D}}$. Since, according to the second term of Eq. (4), volumetric stress σ_{V} depends only on ε_{V} and therefore is the same for all microplanes, Eq. (6) was written as

$$\boldsymbol{\sigma} = \sigma_{\mathrm{V}}\boldsymbol{I} + \frac{3}{2\pi}\int_{\Omega}\sigma_{\mathrm{D}}\boldsymbol{N}\,\mathrm{d}\Omega + \frac{3}{2\pi}\int_{\Omega}\boldsymbol{\sigma}_{\mathrm{T}}\cdot\boldsymbol{T}\,\mathrm{d}\Omega, \tag{8}$$

where $\boldsymbol{I} = 3\boldsymbol{V}$ is the second-order identity tensor (Kronecker delta).

3. Thermodynamic derivation of microplane stresses and equilibrium

The first standard assumption in a thermodynamically consistent constitutive framework is the existence of a free-energy potential per unit mass of material in isothermal conditions, $\Psi^{\mathrm{mac}}(\boldsymbol{\varepsilon}, \boldsymbol{q})$, where $\boldsymbol{q}$ is a given set of internal variables that fully define the state of the material at any point of the loading history. The fundamental assumption for the new, thermodynamically consistent microplane approach is that the macroscopic free energy may be written as the integral of some free energy defined at the microplane level, $\Psi_{\Omega}^{\mathrm{mic}}$:

$$\Psi^{\mathrm{mac}} = \frac{3}{2\pi} \int_{\Omega} \Psi_{\Omega}^{\mathrm{mic}}(\boldsymbol{t}_{\varepsilon}, \boldsymbol{q})\, \mathrm{d}\Omega, \tag{9}$$

where $\boldsymbol{t}_{\varepsilon}$ is the vector collecting the normal and shear strain components for the microplane with normal $\mathbf{n}$.

If the material density is ρ_0, it is a standard procedure (Coleman and Gurtin, 1967; Ilankamban and Krajcinovic, 1987) to obtain the stress conjugate to $\boldsymbol{\varepsilon}$ as the derivative of the free energy per unit volume:

$$\boldsymbol{\sigma} = \frac{\partial[\rho_0 \Psi^{\mathrm{mac}}]}{\partial \boldsymbol{\varepsilon}}. \tag{10}$$

In our case, this formula may be applied to Eq. (9). Using the chain rule of differentiation on the right-hand side, one obtains

$$\boldsymbol{\sigma} = \frac{3}{2\pi} \int_{\Omega} \frac{\partial\left[\rho_0 \Psi_{\Omega}^{\mathrm{mic}}\right]}{\partial \boldsymbol{\varepsilon}}\, \mathrm{d}\Omega = \frac{3}{2\pi} \int_{\Omega} \frac{\partial\left[\rho_0 \Psi_{\Omega}^{\mathrm{mic}}\right]}{\partial \boldsymbol{t}_{\varepsilon}} \cdot \frac{\partial \boldsymbol{t}_{\varepsilon}}{\partial \boldsymbol{\varepsilon}}\, \mathrm{d}\Omega. \tag{11}$$

Assuming that the strain components on the microplane are ε_{N}, and $\boldsymbol{\varepsilon}_{\mathrm{T}}$, and that they are related to the macroscopic strain via the kinematic constraint given by the first and fourth terms of Eq. (1) or Eq. (2), we may expand the two terms of the product inside the integral, obtain the strain derivatives and express

$$\boldsymbol{\sigma} = \frac{3}{2\pi} \int_{\Omega} \frac{\partial\left[\rho_0 \Psi_{\Omega}^{\mathrm{mic}}\right]}{\partial \varepsilon_{\mathrm{N}}} \boldsymbol{N}\, \mathrm{d}\Omega + \frac{3}{2\pi} \int_{\Omega} \frac{\partial[\rho_0 \Psi_{\Omega}]}{\partial \boldsymbol{\varepsilon}_{\mathrm{T}}} \cdot \boldsymbol{T}\, \mathrm{d}\Omega. \tag{12}$$

This equation turns out to be equivalent to Eq. (6) if we define

$$\sigma_{\mathrm{N}} = \frac{\partial\left[\rho_0 \Psi_{\Omega}^{\mathrm{mic}}\right]}{\partial \varepsilon_{\mathrm{N}}}, \qquad \boldsymbol{\sigma}_{\mathrm{T}} = \frac{\partial\left[\rho_0 \Psi_{\Omega}^{\mathrm{mic}}\right]}{\partial \boldsymbol{\varepsilon}_{\mathrm{T}}}. \tag{13}$$

This is actually a consistent definition of the microplane stresses σ_{N} and $\boldsymbol{\sigma}_{\mathrm{T}}$ as the work-conjugate quantities of the microplane strains ε_{N} and $\boldsymbol{\varepsilon}_{\mathrm{T}}$.

If, on the other hand, we consider the formulation *with split*, in which the microplane strains are ε_{V}, ε_{D} and $\boldsymbol{\varepsilon}_{\mathrm{T}}$, developing the second term of Eq. (11) leads to

$$\boldsymbol{\sigma} = \frac{3}{2\pi} \int_{\Omega} \sigma_{\mathrm{V}} \boldsymbol{V}\, \mathrm{d}\Omega + \frac{3}{2\pi} \int_{\Omega} \sigma_{\mathrm{D}} \boldsymbol{D}\, \mathrm{d}\Omega + \frac{3}{2\pi} \int_{\Omega} \boldsymbol{\sigma}_{\mathrm{T}} \cdot \boldsymbol{T}\, \mathrm{d}\Omega \tag{14}$$

with the consistent microplane stresses σ_{V}, σ_{D} and $\boldsymbol{\sigma}_{\mathrm{T}}$ defined as

$$\sigma_{\mathrm{V}} = \frac{\partial\left[\rho_0 \Psi_{\Omega}^{\mathrm{mic}}\right]}{\partial \varepsilon_{\mathrm{V}}}, \qquad \sigma_{\mathrm{D}} = \frac{\partial\left[\rho_0 \Psi_{\Omega}^{\mathrm{mic}}\right]}{\partial \varepsilon_{\mathrm{D}}}, \qquad \boldsymbol{\sigma}_{\mathrm{T}} = \frac{\partial\left[\rho_0 \Psi_{\Omega}^{\mathrm{mic}}\right]}{\partial \boldsymbol{\varepsilon}_{\mathrm{T}}}. \tag{15}$$

4. Discussion

Formulas (12) and (14) obtained by differentiation of the free energy may now be compared to their counterparts in the traditional microplane formulations (7) and (8). The first observation is that, for *the formulation without deviatoric–volumetric split*, the derivation from the free energy leads to identical equations as the original formulation, i.e., the original integral formula for stresses was thermodynamically consistent (although for full consistency of the formulation, the laws ((4), first and second terms) must also be derived from a potential).

In contrast, for *the formulation with split of normal components*, the two derivations exhibit some differences. Comparing Eqs. (8) and (14), we notice the following:

(a) The simple term $\sigma_V \boldsymbol{I}$ in Eq. (8) is replaced by the integral involving the volumetric term (first on the right-hand side) in Eq. (14),

(b) The factor $\boldsymbol{N}$ multiplying σ_D in the deviatoric integral of Eq. (8) is replaced by $\boldsymbol{D} = \boldsymbol{N} - \boldsymbol{V}$ in Eq. (14).

The first difference (a) only vanishes if σ_V may be assumed to be a function of ε_V but not of ε_D and $\boldsymbol{\varepsilon}_T$. In that case, σ_V would be the same for all microplanes and could be taken out of the integral in Eq. (14), that term becoming

$$\frac{3}{2\pi}\int_\Omega \sigma_V \boldsymbol{V}\,\mathrm{d}\Omega = \frac{1}{2\pi}\int_\Omega \sigma_V \boldsymbol{I}\,\mathrm{d}\Omega = \frac{1}{2\pi}\sigma_V \boldsymbol{I}\int_\Omega \mathrm{d}\Omega = \sigma_V \boldsymbol{I}. \tag{16}$$

Since in the first term of Eq. (15), we have defined $\sigma_V = \partial\left[\rho_0 \Psi_\Omega^{\mathrm{mic}}\right]/\partial\varepsilon_V$, having σ_V independent of ε_D and $\boldsymbol{\varepsilon}_T$ implies that the mixed derivatives $\partial^2\left[\rho_0 \Psi_\Omega^{\mathrm{mic}}\right]/\partial\varepsilon_V\partial\varepsilon_D$ and $\partial^2\left[\rho_0 \Psi_\Omega^{\mathrm{mic}}\right]/\partial\varepsilon_V\partial\boldsymbol{\varepsilon}_T$ vanish, but then (because of the remaining definitions ((15), second and third terms)) neither σ_D nor $\boldsymbol{\sigma}_T$ can depend on ε_V either. In this situation, the microplane free energy *must* have the following decoupled form:

$$\Psi_\Omega^{\mathrm{mic}}(\varepsilon_V, \varepsilon_D, \boldsymbol{\varepsilon}_T, \boldsymbol{q}) = \Psi_1^{\mathrm{mic}}(\varepsilon_V, \boldsymbol{q}) + \Psi_2^{\mathrm{mic}}(\varepsilon_D, \boldsymbol{\varepsilon}_T, \boldsymbol{q}). \tag{17}$$

Note that the very assumption of a microplane free energy $\Psi_\Omega^{\mathrm{mic}}$ which depends on the strains on the same microplane exclusively may be in itself quite restrictive. For instance, the latest practical formulation for concrete M4 (Bažant et al., 2000b), and also its predecessor M3 (Bažant et al., 1996a), use a procedure to calculate σ_V and σ_D on each microplane which makes them actually dependent on deviatoric strains ε_D on *all other* microplanes. That was a way to combine the advantages of the model without split in tension with those of the split in compression, and allowed a much better fit of experimental data for concrete. That same feature, however, makes those formulations more general than the thermodynamic framework considered in this paper, and for them, the question of work conjugacy must be addressed in a different way (Bažant et al., 2000b). In contrast, earlier versions of the model which, in retrospective we can call *M*1 (Bažant and Oh, 1983; Bažant and Gambarova, 1984) and *M*2 (Bažant and Prat, 1988a,b; Carol et al., 1992), did conform to assumption (9), and most of them actually also to Eq. (17). This might not be apparent in some cases, in which the microplane law for the shear components involved some form of dependence on the volumetric strain in order to introduce the frictional effect of hydrostatic pressure on the deviatoric behavior. Nevertheless, the nature of that dependence is that of a shear yield limit that depends on normal stress, while unloading properties (which relate to stored energy) remain uncoupled. For this reason, this effect may be in general introduced via the history variables $\boldsymbol{q}$, with the practical consequence that, for all those formulations included in the framework, difference (a) is only apparent and does not imply real inconsistency.

More essential, however, is the difference (b). In effect, by developing the second integral in Eq. (14), we can write

$$\int_{\Omega} \sigma_{\mathrm{D}} \boldsymbol{D} \,\mathrm{d}\Omega = \int_{\Omega} \sigma_{\mathrm{D}} \boldsymbol{N} \,\mathrm{d}\Omega - \boldsymbol{V} \int_{\Omega} \sigma_{\mathrm{D}} \,\mathrm{d}\Omega. \tag{18}$$

The second term on the right-hand side vanishes only if the average deviatoric stress is zero, and this is actually only satisfied for a very narrow class of models. The most important member of this class is isotropic linear elasticity, which is described by the microplane free-energy potential

$$\rho_0 \Psi_{\Omega}^{\mathrm{mic}}(\varepsilon_{\mathrm{V}}, \varepsilon_{\mathrm{D}}, \boldsymbol{\varepsilon}_{\mathrm{T}}) = \tfrac{3}{2} K \varepsilon_{\mathrm{V}}^2 + G \left[\varepsilon_{\mathrm{D}}^2 + |\boldsymbol{\varepsilon}_{\mathrm{T}}|^2 \right], \tag{19}$$

where K is the macroscopic bulk modulus and G is the macroscopic shear modulus of elasticity. The mean value of ε_{D} over all the microplanes is always zero, and as $\sigma_{\mathrm{D}} = \partial \left[\rho_0 \Psi_{\Omega}^{\mathrm{mic}} \right] / \partial \varepsilon_{\mathrm{D}} = 2G\varepsilon_{\mathrm{D}}$, the mean value of σ_{D} is zero as well. Note, however, that as soon as any nonlinear behavior is considered, this condition is immediately violated. Exception to this rule would only take place in a two-dimensional version of the microplane model (with integration over a unit semicircle rather than a unit hemisphere), provided that the deviatoric stress depends only on the deviatoric strain, and that the response is symmetric in tension and in compression. In other words, the deviatoric law must be such that if a deviatoric strain evolution $\varepsilon_{\mathrm{D}}(t)$ produces stress $\sigma_{\mathrm{D}}(t)$, then $-\varepsilon_{\mathrm{D}}(t)$ produces $-\sigma_{\mathrm{D}}(t)$. This condition is satisfied for example by nonlinear hyperelastic models with a deviatoric law of the form,

$$\sigma_{\mathrm{D}} = f_{\mathrm{D}}(\varepsilon_{\mathrm{D}}), \qquad f_{\mathrm{D}}(-\varepsilon_{\mathrm{D}}) = -f_{\mathrm{D}}(\varepsilon_{\mathrm{D}}), \tag{20}$$

i.e., f_{D} is an odd function of ε_{D}. In two dimensions, each microplane may be associated with the perpendicular one, and the deviatoric strains on these associated microplanes have the same magnitude and opposite signs. If the deviatoric law is symmetric, the resulting deviatoric stresses on the associated microplanes also have the same magnitude and opposite signs, and their mean value over all the microplanes is zero.

Unfortunately, the precedent reasoning cannot be extended to three dimensions or to general (nonodd) functions $f_{\mathrm{D}}(\varepsilon_{\mathrm{D}})$, and therefore, the conclusion is that early practical formulations of concrete microplane models with volumetric–deviatoric split did not satisfy thermodynamic consistency. In those formulations, a well-defined energy potential *did not exist*, and in general, the microplane stresses σ_{V} and σ_{D} *were not* the work-conjugates of their counterparts ε_{V} and ε_{D}. Also, the possibility existed of spurious dissipation/generation of energy taking place for appropriately designed load cycles, as shown in the example developed in Section 5. It is finally noted that the thermodynamically consistent formula (14) may also be derived from the PVW but only if the contribution of the normal microplane stresses to the virtual work in Eq. (5) is rewritten as $\int_{\Omega} [\sigma_{\mathrm{V}} \,\delta\varepsilon_{\mathrm{V}} + \sigma_{\mathrm{D}} \,\delta\varepsilon_{\mathrm{D}}] \,\mathrm{d}\Omega$.

5. Example of spurious dissipation

To illustrate the problem, consider a model with microplane laws in secant format proposed in Bažant and Prat (1988a), which was developed later in a damage format in Kuhl et al. (1998). In such formulation, the microplane constitutive equations read

$$\sigma_{\mathrm{V}} = [1 - d_{\mathrm{V}}] 3K\varepsilon_{\mathrm{V}}, \qquad \sigma_{\mathrm{D}} = [1 - d_{\mathrm{D}}] 2G\varepsilon_{\mathrm{D}}, \qquad \boldsymbol{\sigma}_{\mathrm{T}} = [1 - d_{\mathrm{T}}] 2G\boldsymbol{\varepsilon}_{\mathrm{T}}, \tag{21}$$

where d_{V}, d_{D} and d_{T} are scalar damage parameters, initially set to zero. The simplest assumption is that d_{V} depends only on the history of ε_{V}, d_{D} depends only on the history of ε_{D}, and d_{T} only on the history of $|\boldsymbol{\varepsilon}_{\mathrm{T}}|$. Parameter d_{V} is then the same on all microplanes while d_{D} and d_{T} in general vary as functions of the microplane orientation.

Substituting the microplane laws into the traditional stress-evaluation formula (8) and using Eq. (2) yields

$$\begin{aligned}\boldsymbol{\sigma} &= [1-d_V]3K\boldsymbol{I}\varepsilon_V + \frac{3G}{\pi}\int_\Omega [1-d_D]\boldsymbol{N}\varepsilon_D\,d\Omega + \frac{3G}{\pi}\int_\Omega [1-d_T]\boldsymbol{T}^T\cdot\boldsymbol{\varepsilon}_T\,d\Omega \\ &= [1-d_V]K\boldsymbol{I}\otimes\boldsymbol{I}:\boldsymbol{\varepsilon} + \frac{3G}{\pi}\int_\Omega [1-d_D]\boldsymbol{N}\otimes\boldsymbol{D}\,d\Omega:\boldsymbol{\varepsilon} + \frac{3G}{\pi}\int_\Omega [1-d_T]\boldsymbol{T}^T\cdot\boldsymbol{T}\,d\Omega:\boldsymbol{\varepsilon} \\ &= \mathbf{E}:\boldsymbol{\varepsilon},\end{aligned} \tag{22}$$

where $\boldsymbol{T}^T$ is a third-order tensor with components $T^T_{ijr} = T_{rij}$, and

$$\mathbf{E} = [1-d_V]K\boldsymbol{I}\otimes\boldsymbol{I} + \frac{3G}{\pi}\int_\Omega [1-d_D]\boldsymbol{N}\otimes\boldsymbol{D}\,d\Omega + \frac{3G}{\pi}\int_\Omega [1-d_T]\boldsymbol{T}^T\cdot\boldsymbol{T}\,d\Omega \tag{23}$$

is the secant macroscopic stiffness tensor. Due to the presence of the term,

$$\int_\Omega [1-d_D]\boldsymbol{N}\otimes\boldsymbol{D}\,d\Omega = \int_\Omega [1-d_D]\boldsymbol{D}\otimes\boldsymbol{D}\,d\Omega + \boldsymbol{V}\otimes\int_\Omega [1-d_D]\boldsymbol{D}\,d\Omega, \tag{24}$$

the stiffness tensor $\mathbf{E}$ in general does not exhibit major symmetry. Only in the case of isotropic damage, we can take the factor $1-d_D$ out of the integral and write

$$\boldsymbol{V}\otimes\int_\Omega [1-d_D]\boldsymbol{D}\,d\Omega = [1-d_D]\boldsymbol{V}\otimes\int_\Omega \boldsymbol{D}\,d\Omega = \boldsymbol{O} \tag{25}$$

since

$$\int_\Omega \boldsymbol{D}\,d\Omega = \int_\Omega \boldsymbol{N}\,d\Omega - \boldsymbol{V}\int_\Omega d\Omega = \frac{2\pi}{3}\boldsymbol{I} - \boldsymbol{V}2\pi = \boldsymbol{O}. \tag{26}$$

The lack of major symmetry would not necessarily be in contradiction to the laws of thermodynamics if it were caused by frictional phenomena. However, this is not the case here. During unloading and reloading below the maximal previously reached strain level, the damage parameters remain constant and the material responds as a linear elastic one with stiffness $\mathbf{E}$. The lack of major symmetry then implies that no elastic potential can exist, and the total work over a closed cycle is in general not zero. For certain loading cycles, energy is consumed, and for others, it is extracted from the material (without changing the internal variables).

A loading cycle generating energy can be constructed as follows: For simplicity, we assume that the shear microplane stresses do not arise, i.e., we set $d_T = 1$ on all microplanes. The other damage parameters are initially zero. Now we apply a strain cycle consisting of four steps:

1. application of purely deviatoric strain, $\mathbf{e}_0$;
2. application of additional volumetric strain, $\varepsilon_0\boldsymbol{I}$;
3. removal of the deviatoric strain; and
4. removal of the volumetric strain.

When this cycle is applied for the first time, the damage parameters grow. For a typical nonsymmetric deviatoric law, the deviatoric damage parameter d_D is larger on microplanes that experience deviatoric tension than on those that are under deviatoric compression of the same magnitude (Fig. 1a). When the same cycle is repeated, all microplane strain components remain within the previously reached limits, and so damage does not grow any more. The total work during this closed cycle should be zero, but we will show that this is not the case.

In the first step (of the second cycle), the volumetric strain remains zero, and the deviatoric microplane strain $\varepsilon_D = \boldsymbol{D}:\mathbf{e}_0$ has a zero mean over all the microplanes. The macroscopic stress evaluated according to Eq. (8),

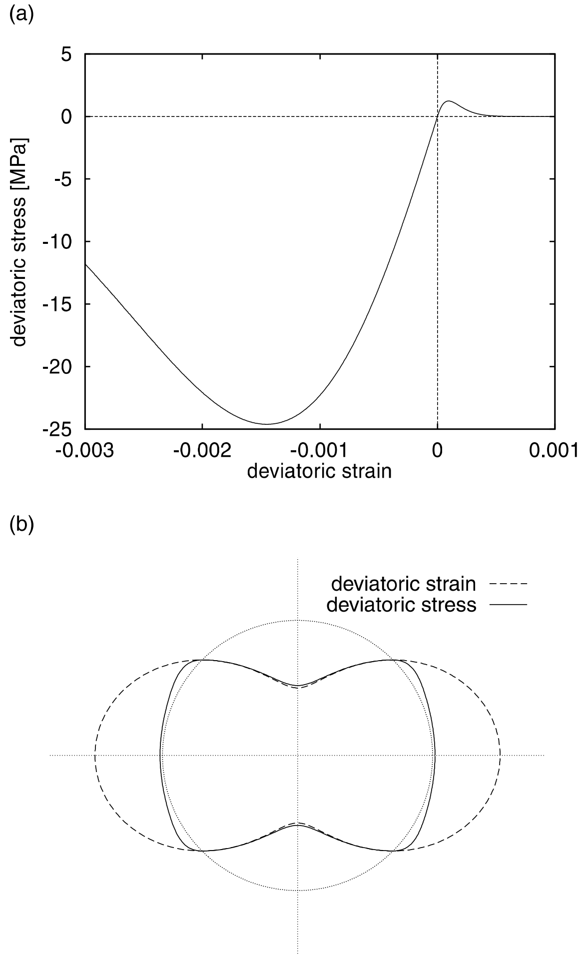

Fig. 1. (a) Deviatoric microplane law and (b) distribution of deviatoric microplane strain and stress.

$$\boldsymbol{\sigma} = \frac{3}{2\pi}\int_{\Omega} \sigma_{\mathrm{D}} \boldsymbol{N}\,\mathrm{d}\Omega \tag{27}$$

has the mean value

$$\sigma_{\mathrm{m}} = \boldsymbol{V} : \boldsymbol{\sigma} = \frac{3}{2\pi}\int_{\Omega} \sigma_{\mathrm{D}} \boldsymbol{V} : \boldsymbol{N}\,\mathrm{d}\Omega = \frac{1}{2\pi}\int_{\Omega} \sigma_{\mathrm{D}}\,\mathrm{d}\Omega \tag{28}$$

because $\boldsymbol{V} : \boldsymbol{N} = \delta_{ij} n_i n_j / 3 = n_i n_i / 3 = 1/3$. The last term represents the mean value of σ_{D} over all the microplanes,

$$\bar{\sigma}_{\mathrm{D}} = \frac{1}{2\pi}\int_{\Omega} \sigma_{\mathrm{D}}\,\mathrm{d}\Omega = \frac{G}{\pi}\int_{\Omega} [1 - d_{\mathrm{D}}]\varepsilon_{\mathrm{D}}\,\mathrm{d}\Omega, \tag{29}$$

which is negative due to the fact that damage on planes with $\varepsilon_{\mathrm{D}} > 0$ is larger than on those with $\varepsilon_{\mathrm{D}} < 0$ (Fig. 1b). This means that the macroscopic stress tensor after the first step has the volumetric part $\sigma_{\mathrm{m}} = \bar{\sigma}_{\mathrm{D}}$ and the deviatoric part

Table 1
Loading cycle

Step	$\boldsymbol{\varepsilon}$	σ_V	σ_D	$\boldsymbol{\sigma}$	$\int \boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon}$
0	$\mathbf{o}$	0	0	$\mathbf{o}$	$\mathbf{o}$
1	$\mathbf{e}_0$	0	$[1-d_D]2G\boldsymbol{D} : \mathbf{e}_0$	$\mathbf{s}_0 + \bar{\sigma}_D \boldsymbol{I}$	$\mathbf{s}_0 : \mathbf{e}_0/2$
2	$\mathbf{e}_0 + \varepsilon_0 \boldsymbol{I}$	$[1-d_V]3K\varepsilon_0$	$[1-d_D]2G\boldsymbol{D} : \mathbf{e}_0$	$\mathbf{s}_0 + [\bar{\sigma}_D + \sigma_0]\boldsymbol{I}$	$3[\bar{\sigma}_D + \sigma_0/2]\varepsilon_0$
3	$\varepsilon_0 \boldsymbol{I}$	$[1-d_V]3K\varepsilon_0$	0	$\sigma_0 \boldsymbol{I}$	$-\mathbf{s}_0 : \mathbf{e}_0/2$
4	$\mathbf{o}$	0	0	$\mathbf{o}$	$-3\sigma_0\varepsilon_0/2$
Total dissipation					$\sum = 3\bar{\sigma}_D\varepsilon_0$

$$\mathbf{s}_0 = \boldsymbol{\sigma} - \sigma_m \boldsymbol{I} = \frac{3}{2\pi} \int_\Omega [\sigma_D - \bar{\sigma}_D] \boldsymbol{N} \, \mathrm{d}\Omega. \tag{30}$$

In the second step, the volumetric stress is increased by $\sigma_0 = [1 - d_V]3K\varepsilon_0$ while the deviatoric part of the stress tensor remains unchanged. In the third step, the deviatoric stress is removed, and the volumetric stress changes by $-\bar{\sigma}_D$. Finally, after the fourth step, all stresses disappear and the cycle is closed. Individual steps of the cycle are summarized in Table 1.

Now, let us look at the total work during the cycle, $\oint \boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon}$. The work done by the deviatoric stresses during the first step is canceled by the work done during the third step, and the work done by σ_0 during the second step is canceled by the work done during the fourth step. However, the work done by $\bar{\sigma}_D$ during the second step has no counterpart since $\bar{\sigma}_D$ is not present during the fourth step. Consequently, the total work done during the cycle is $\bar{\sigma}_D \boldsymbol{I} : \varepsilon_0 \boldsymbol{I} = 3\bar{\sigma}_D \varepsilon_0 \neq 0$. The sign of this work depends on ε_0, since $\bar{\sigma}_D$ is always negative. For positive ε_0, we obtain a negative work over the cycle, which means that the material supplies energy to its environment without changing its internal state.

6. Concluding remarks

A new, simple, thermodynamically-consistent framework is presented for the formulation of microplane models. The main assumption is that the macroscopic free energy may be obtained as the integral over all microplane orientations of a microplane free-energy function, which depends on the microplane strains and the internal variables. This assumption does not contradict most of the early versions of microplane models for concrete with and without split of normal components (M1 and M2), but leaves out the more recent M3 and M4 models, for which the free energy of the various microplanes may not be written in a decoupled form.

The new formulation leads to a consistent definition of the microplane stresses which are conjugate to the microplane strains, and to the integral form of the micro–macro equilibrium equation which applies to those stresses.

A comparison with the previous microplane models not precluded by the new formulation leads to the conclusion that, while the earliest model without split (M1) was correct, the following version of microplane model with the split of normal components (M2) cannot be guaranteed to be thermodynamically consistent. In that case, microplane stresses σ_V and σ_D are not necessarily work conjugates to their strain counterparts ε_V and ε_D, and neither are in general their sums $\sigma_N = \sigma_V + \sigma_D$ and $\varepsilon_N = \varepsilon_V + \varepsilon_D$. The integral micro–macro relation for stresses does not coincide either with the one obtained from the thermodynamic derivation, and the model cannot be guaranteed to be free of spurious dissipation or generation of energy. Models with "symmetric" laws for the normal deviatoric component (in the sense of symmetric behavior in tension and in compression) are less sensitive to this problem.

In a companion paper "Part II", the new thermodynamic derivation is developed further by applying standard concepts such as the Coleman method and Clausius–Duhem inequality at both microplane and macroscopic levels, and the resulting equations are illustrated with two example formulations of damage and plasticity (Kuhl et al., 2000).

Acknowledgements

Partial support from DGICYT (Madrid, Spain) under research grants PB95-0771 and PB96–0500 is gratefully acknowledged. The second author received travel support through grant No. GRQ93–3012 from CUR (Generalitat de Catalunya, Barcelona, Spain), and his work at EPF Lausanne was supported by CTI (Swiss Commission for Technology and Innovation). The last author wishes to thank for the partial support from the US National Science Foundation under Grant CMS-9732791.

References

Batdorf, S.B., Budiansky, B., 1949. A mathematical theory of plasticity based on the concept of slip. Technical note 1871, National Advisory Committee for Aeronautics.

Bažant, Z.P., Oh, B.H., 1983. Microplane model for fracture analysis of concrete structures. Proc. Symp. Interact. Non-Nucl. Munitions Struct., US Air Force Academy, Springs, Co, pp. 49–55.

Bažant, Z.P., Gambarova, P., 1984. Crack shear in concrete: crack band microplane model. ASCE J. Struct. Engng. 110, 2015–2035.

Bažant, Z.P., Oh, B.H., 1985. Microplane model for progressive fracture of concrete and rock. ASCE J. Engng. Mech. 111, 559–582.

Bažant, Z.P., Prat, P.C., 1987. Creep of anisotropic clay: new microplane model. ASCE J. Engng. Mech. 103 (7), 1050–1064.

Bažant, Z.P., Prat, P.C., 1988a. Microplane model for brittle-plastic material: I. Theory. ASCE J. Engng. Mech. 114 (10), 1672–1688.

Bažant, Z.P., Prat, P.C., 1988b. Microplane model for brittle-plastic material: II. Verification. ASCE J. Engng. Mech. 114 (10), 1689–1702.

Bažant, Z.P., Xiang, Y., Prat, P.C., 1996a. Microplane model for concrete: I. Stress–strain boundaries and finite strain. ASCE J. Engng. Mech. 122 (3), 245–262.

Bažant, Z.P., Xiang, Y., Adley, M.D., Prat, P.C., Akers, S.A., 1996b. Microplane model for concrete: II. Data delocalization and verification. ASCE J. Engng. Mech. 122 (3), 263–268.

Bažant, Z.P., Adley, M.D., Carol, I., Jirásek, M., Akers, S.A., Cargile, J.D., Rohani, B., Caner, F.C., 2000a. Large-strain generalization of microplane constitutive model for concrete. ASCE J. Engng. Mech., in press.

Bažant, Z.P., Caner, F.C., Carol, I., Adley, M.D., Akers, S.A., 2000b. Microplane model M4 for concrete: I. formulation with work-conjugate deviatoric stress. ASCE J. Engng. Mech., in press.

Carol, I., Bažant, Z.P., 1997. Damage and plasticity in microplane theory. Int. J. Solids Struct. 34 (29), 3807–3835.

Carol, I., Bažant, Z.P., Prat, P.C., 1991. Geometric damage tensor based on microplane model. ASCE J. Engng. Mech. 117, 2429–2448.

Carol, I., Prat, P.C., Bažant, Z.P., 1992. New explicit microplane model for concrete: theoretical aspects and numerical implementation. Int. J. Solids Struct. 29 (9), 1173–1191.

Carol, I., Jirásek, M., Bažant, Z.P., Steinmann, P., 1998. New thermodynamic approach to microplane model with application to finite deformations. Technical report PI-145, Int. Center Numer. Meth. Engng. (CIMNE), ETSECCPB-UPC, E-08034 Barcelona, Spain.

Cofer, W.F., 1992. Implementation of the nonlocal microplane concrete model within an explicit dynamic finite element program. Appl. Mech. Rev. 45 (3), S132–S139.

Cofer, W.F., Kohut, S.W., 1994. A general nonlocal microplane concrete material model for dynamic finite element analysis. Comput. and Struct. 53 (1), 189–199.

Coleman, B.D., Gurtin, M.E., 1967. Thermodynamics with internal state variables. J. Chem. Phys. 47, 597.

Fichant, S., 1996. Damage and Induced Anisotropy in Structural Concrete. Ph.D. thesis, Laboratoire de Mechanique et Technologie, Ecole Normale Superieure, 61, av. du Pdt. Wilson, 94235 Cachan Cedex, France (In French).

Ilankamban, R., Krajcinovic, D., 1987. A constitutive theory for progressively deteriorating brittle solids. Int. J. Solids Struct. 23 (11), 1521–1534.

Kuhl, E., Carol, I., Steinmann, P., 2000. A thermodynamically consistent approach to microplane theory. Part II. Dissipation and inelastic constitutive modelling. Int. J. Solids Struct. 38 (17), 2933–2952.

Kuhl, E., Ramm, E., de Borst, R., 1998. Anisotropic gradient damage with the microplane model. In: Mang, H., Bićanić, N., de Borst, R. (Eds.), Comput. Model. Concrete Struct. Balkema, Badgastein, Austria, pp. 103–112.

Mohr, O., 1900. Welche Umstande bendingen der Bruch und der Elastizitätsgrenze des Materials. Z. Vereins Deutscher Ingenieure 44, pp. 1–12.

Ožbolt, J., Bažant, Z.P., 1992. Microplane model for cyclic triaxial behavior of concrete. ASCE J. Engng. Mech. 118 (7), 1365–1386.

Ožbolt, J., Bažant, Z.P., 1996. Numerical smeared fracture analysis: nonlocal microcrack interaction approach. Int. J. Num. Meth Engng. 39 (4), 635–661.

Pande, G.N., Sharma, K.G., 1983. Multilaminate model of clays – a numerical evaluation of the influence of rotation of principal axes. ASCE J. Engng. Mech. 109 (7), 397–418.

Taylor, G.I., 1938. Plastic strain in metals. J. Inst. Metals 62, 307–324.

Zienkiewicz, O.C., Pande, G.N., 1977. Time-dependent multi-laminate model of rocks – a numerical study of deformation and failure of rock masses. Int. J. Num. Anal. Meth. Geomech. 1, 219–247.